PROPERTIES OF EIGEN VALUES

1. If λ is the eigen value of A. Then λ^m is the eigen value of A^m

Proof: If λ is the eigen value of A, then by definition we have

$$AX = \lambda X \quad \text{—— (1)}$$

To prove λ^m is the eigen value of A^m

i.e. $$A^m X = \lambda^m X \quad \text{—— (2)}$$

We can prove this by induction method

Step 1. put $m = 1$ in (2)

$\Rightarrow$ $AX = \lambda X$ which is true.

Hence Result is true for $m=1$

Step 2. put $m = k$ in (2)

$\Rightarrow$ $A^k X = \lambda^k X$ — Supposing result is true for $m=k$

Step 3. To prove that result is true for $m=k+1$

i.e To prove $A^{k+1} X = \lambda^{k+1} X$

Now $A^{K+1}X = A(A^K X)$

$= A(\lambda^K X)$ (from Step 2.)

$= \lambda^K (AX)$

$= \lambda^K \cdot (\lambda X)$ (from ①)

$A^{K+1}X = \lambda^{K+1} X$

Hence the result is true for all values of m.

Hence by <u>Induction method.</u>

λ^m is the eigen value of A^m.

<u>Result</u> : If λ is the eigen value of A. then $K\lambda$ is the eigen value of KA.

$AX = \lambda X$

Multiplying both sides by K

$\underline{KA}X = \underline{K\lambda}X$

Hence $K\lambda$ is the eigen value of KA.

2. If λ is the eigen value of A. then $\frac{1}{\lambda}$ is the eigen value of A^{-1}

Proof: If λ is the eigen value of A, the by definition

$$AX = \lambda X \quad — ①$$

pre multiplying both sides by A^{-1}

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$IX = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

OR $$\boxed{A^{-1}X = \frac{1}{\lambda}X}$$

Hence $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

3. If λ is the eigen value of A. then $\frac{|A|}{\lambda}$ is the eigen value of adj A.

Proof. By definition we have

$$AX = \lambda X \quad — ①$$

Also, $A^{-1}X = \frac{1}{\lambda}X$ (Proved in property 2)

$$\Rightarrow \frac{adj A}{|A|} X = \frac{1}{\lambda} X$$

$$adj A \, X = \frac{|A|}{\lambda} X$$

Hence $\frac{|A|}{\lambda}$ is the eigen value of adj A.

<u>Note.</u> This result can also be proved by pre multiplying ① by adj A. And then by using the definition $adj A = |A| A^{-1}$.

4. A Square matrix A and its transpose A^t have the Same Set of eigen values.

<u>Proof</u>: The above result Can be proved as by proving that A and A^t will have the Same Characteristic equations / characteristic polynomial.

Let the characteristic equation of A^t is

$$|A^t - \lambda I| = 0$$

where λ, is the eigen Value of A^t

or $|A^t - \lambda I^t| = 0$

or $|(A - \lambda I)^t| = 0$

or $|A - \lambda I| = 0$ as $|A| = |A^t|$

which is the characteristic equation for A too, with eigen value λ.

Hence proved

5. At least one Latent root/ Eigen value of Every Singular matrix is Zero.

<u>Proof.</u> Given A is Singular matrix

$\Rightarrow \quad |A| = 0$

$\Rightarrow \quad |A - 0| = 0$

$\Rightarrow \quad |A - 0I| = 0$

$\Rightarrow \lambda = 0$, Satisfies the equation $|A - \lambda I| = 0$

$\Rightarrow \lambda = 0$ is the Latent root of A.

<u>Imp PTU EXAMPLE</u>

Show that, if Zero is the Eigen value of A, then it is Singular.

<u>Sol:</u> If A is the given matrix, then its Characteristic equation is given by

$$|A - \lambda I| = 0$$

$$|A - 0I| = 0 \quad (\text{If } \lambda = 0)$$

$\Rightarrow \quad |A| = 0$

$\Rightarrow$ A is a Singular matrix.

* Show that the eigen values of a triangular matrix are just the diagonal elements of the matrix.

<u>Solution</u> Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}_{n\times n}$

be an upper triangular matrix of order n.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n\times n}$$

$$[A-\lambda I] = \begin{bmatrix} a_{11}-\lambda & \cdots & & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Eigen value of A are given by

$$|A-\lambda I| = 0$$

or $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0$

$\Rightarrow \lambda = a_{11}, a_{22} \cdots a_{nn}$, which are just the diagonal elements of A.

Similary result Can be proved for a lower triangular Matrix.

Hence proved.